Discussion Summary: on Finding a Gravity Description of Heavy Ion Collisions

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I. MOTIVATION

Heavy-ion collisions are thought to give rise to a strongly-coupled QGP; one of the goals of this program is to better understand how to describe this collision process, from the moment of impact to the onset of hadronization, using gravity in AdS. The tool that enables this description is the AdS/CFT correspondence. This conjectured duality, in the regime where[1] the AdS length scale L is much larger than the string scale l_s, relates Einstein gravity on AdS_5 to $\mathcal{N}=4$ super-Yang-Mills (at strong coupling) defined on its boundary. This duality approach to the problem of describing heavy-ion collisions is complementary to perturbative $\mathcal{N}=4$ SYM calculations (which assume weak coupling), and to lattice QCD calculations (which are usually formulated with a Euclidean action, and are thus not well-adapted to studying dynamical properties of the plasma).

From the gravity perspective, one of the most important entries in the AdS/CFT dictionary is:

$$\langle T_{\mu\nu}\rangle_{\mathrm{CFT}} = \lim_{q\to 0}\frac{1}{q^2}{}^{(q)}T_{\mu\nu}. \tag{1}$$

Here, q is a smooth positive scalar with a simple zero at the AdS boundary, and ${}^{(q)}T_{\mu\nu}$ is the Brown-York quasi-local stress tensor [1]. For asymptotically AdS_5 spacetimes, the quasi-local stress tensor defined on a $q=$ const. time-like hypersurface ∂M_q was constructed in [2], and (neglecting a constant term due to Casimir contributions) is given by

$${}^{(q)}T^0_{\mu\nu} = \frac{1}{8\pi}\left({}^{(q)}\Theta_{\mu\nu} - {}^{(q)}\Theta\Sigma_{\mu\nu} - \frac{3}{L}\Sigma_{\mu\nu} + {}^{(q)}G_{\mu\nu}\frac{L}{2}\right). \tag{2}$$

Here, ${}^{(q)}\Theta_{\mu\nu} = -\Sigma^{\alpha}{}_{\mu}\Sigma^{\beta}{}_{\nu}\nabla_{(\alpha}S_{\beta)}$ is the extrinsic curvature of the time-like surface ∂M_q, S^μ is a space-like, outward pointing unit vector normal to the surface ∂M_q, $\Sigma_{\mu\nu}\equiv g_{\mu\nu}-S_\mu S_\nu$ is the induced 4-metric on ∂M_q, ∇_α is the covariant derivative operator, and ${}^{(q)}G_{\mu\nu}$ is the Einstein tensor associated with $\Sigma_{\mu\nu}$. The last two terms in (2) are counterterms designed to exactly cancel the divergent boundary behavior of the first two terms of (2) evaluated in pure AdS_5.

[1] Note that in Planck units, $L^4/{l_s}^4 \sim N$, where the gauge theory described by the gravity dual has gauge group $SU(N)$; thus, a necessary condition for a classical gravity description is $N\to\infty$.

II. INITIAL DATA

The starting point of any gravity description is initial data: the metric and matter fields defined on an initial space-like or null hypersurface. A crucial question is whether initial data specified on the gravity side is relevant to an interesting physical process on the boundary field theory (i.e. some process that reasonably closely models a heavy-ion collision).

III. SYMMETRIES

In the Bjorken picture of heavy-ion collisions, the solution has rotational symmetry and translational symmetry in the transverse plane $x_1 - x_2$, and boost-invariance along the beam-line direction x_3. The generators of the Lie algebra underlying this $ISO(2)\times SO(1,1)$ symmetry group can be geometrically described as Killing vectors

$$\begin{aligned}\xi &= x^1\partial_{x^1} - x^2\partial_{x^1}\\ \xi &= \partial_{x^1}\\ \xi &= \partial_{x^2}\\ \xi &= x^3\partial_t + t\partial_{x^3}.\end{aligned} \tag{3}$$

A toy gravity model for this kind of setup is of two colliding shockwaves in AdS. In Fefferman-Graham coordinates adapted to the Poincaré patch of global AdS, the metric of two shockwaves reads[2]

$$\begin{aligned}g_{\mu\nu}dx^\mu dx^\nu &= \frac{1}{z^2}\left[dz^2 - 2dx^+dx^- + \mu_1 z^4\delta(x^-)(dx^-)^2\right.\\ &\quad\left.+\mu_2 z^4\delta(x^+)(dx^+)^2\right].\end{aligned} \tag{4}$$

Notice, however, that such shock-waves solutions do *not* exhibit the boost-invariant feature of the Bjorken picture **(many thanks to Romuald Janik for demonstrating this)**. To see why this is so, simply notice that the $\mu_1 z^4\delta(x^-)(dx^-)^2 + \mu_2 z^4\delta(x^+)(dx^+)^2$ piece of the metric, when written in terms of τ,η using $x^\pm = \tau\exp(\pm\eta)/\sqrt{2}$, has η-dependence (recall that boosts act via $\eta\to\eta+const$).

It is also interesting to think about how to go beyond the Bjorken picture, particularly by relaxing the translational invariance in the transverse plane, generated by

[2] Here, we also define light-cone coordinates $x^\pm = (t+z)/\sqrt{2} = \tau\exp(\pm\eta)/\sqrt{2}$ where t,z are the usual Poincaré coordinates on AdS and τ,η are the proper time and spacetime rapidity coordinates.

the Killing vectors $\xi = \partial_{x^1}$ and $\xi = \partial_{x^2}$. Relaxing this symmetry may better capture the finite extent of the colliding nuclei, the $\gamma = 100$ factor nonewithstanding (with translational invariance, these nuclei are effectively modeled as sheets of infinite extent in the transverse plane $x^1 - x^2$). One possibility is by relaxing the $ISO(2)$ symmetry of rotations and translations in the transverse plane, to a conformal $SO(3)$ symmetry as discussed in [3–6]. Under this relaxed symmetry, rotations around the axis of symmetry of the Minkowski space flow are retained, as they form an $SO(2)$ subgroup of the conformal $SO(3)$ symmetry. The rest of this conformal $SO(3)$ is composed of special conformal transformations, corresponding to conformal Killing vectors of Minkowski space.

IV. THERMALIZATION, HYDRODYNAMICS, ISOTROPIZATION

There are three physical times that are often cited in gravity calculations motivated by heavy-ion collisions. These are: thermalization time, isotropization time, and the time it takes for the dual CFT stress tensor to behave like that of a fluid

$$T_{\mu\nu} = (\epsilon + P)u_\mu u_\nu + P g_{\mu\nu} - 2\eta\sigma_{\mu\nu} + \Pi_{\mu\nu}, \qquad (5)$$

where we have introduced the (symmetric, traceless) shear tensor

$$\sigma^{\mu\nu} = \perp^{\mu\alpha}\perp^{\nu\beta}\nabla_{(\alpha}u_{\beta)} - \frac{1}{d-1}\nabla_\alpha u^\alpha \perp^{\mu\nu}, \qquad (6)$$

and subsumed all higher-order terms under $\Pi_{\mu\nu}$. In the above, u_μ is the fluid velocity, and ϵ, P are the rest-frame energy density and pressure of the fluid. The isotropization time is definitely distinct and larger than the two others, but the consensus appeared to be that thermalization time is to be identified with the hydrodynamic time i.e. we say that the solution has "thermalized" when the CFT stress tensor can be parametrized in the form (5).

V. CFT TIME, BULK ADS TIME

The relation between CFT time and the notion of time in the bulk AdS spacetime is usually expressed in terms of ingoing null geodesics emanating from the boundary in the case of configurations such the global AdSBH with a static horizon, though it is not immediately clear how to extend this in the more dynamical context. A proposal for this extension comes from the fluid/gravity correspondence, where local neighborhoods of the boundary are extended along radial null geodesics into the bulk spacetime. This suggests a local mapping between the boundary and the bulk spacetime (**thanks to Gary Horowitz for emphasizing the locality of this proposal**). This mapping can, for example, be used to define the entropy associated with a region of the boundary CFT dual to a bulk AdSBH: simply trace the ingoing radial null geodesics from that boundary region into the bulk, and read off the area element of the horizon that they intersect and compute $S = A/4$ (in geometric units where the gravitational constant is $G = 1$).

More general perscriptions for computing entropy have been proposed, wherein the surface whose surface area is to be computed need not be the event horizon. In fact, the apparent horizon seems to be a more useful object to keep track of, though one needs to keep in mind that apparent horizons depend on ones choice of bulk coordinates (**thanks to Michal Heller for pointing this out**).

VI. NUMERICAL SCHEMES FOR GRAVITY SIMULATIONS

The three different schemes most widely used in numerical relativity are: characteristics, BSSN, generalized harmonic. We proceed in order of the schemes least-known to the author, to those best-known.

A. Characteristic Scheme

The characteristic evolution scheme is named for the null-coordinates with which the metric ansatz is expressed, and the evolution carried out. Such coordinates are well-adapted to describing such colliding plane waves. This method simplifies the treatment of the AdS_5 boundary. It also greatly simplified the problem of finding initial data, since the momentum constraints are trivially satisfied on any null hypersurface.

B. BSSN Scheme

The BSSN scheme is based on the ADM *3 + 1* decomposition of the Einstein equations, coupled with a conformal decomposition. Here, we simply state the *3 + 1* system of equations, which contain terms involving the spatial energy-momentum tensor, denoted $\mathbf{S} = S_{ij} dx^i \otimes dx^j$ with components $S_{ij} = h_i{}^a h_j{}^b T_{ab}$, terms involving the energy density, denoted $\rho = n^a n^b T_{ab}$, and terms involving the momentum density, denoted $\mathbf{j} = j_i dx^i$ with components $j_i = -h_i{}^a n^b T_{ab}$. The second fundamental form (extrinsic curvature) of each spatial hypersurface is given by $\mathbf{K} = K_{ij} dx^i \otimes dx^j$ with components $K_{ij} = -h_i{}^a h_j{}^b (\nabla_a n)_b = -\frac{1}{2} L_{\mathbf{n}} h_{ij}$. Contractions of the second fundamental form are calculated with respect to the induced 3-metric, so that $K = K^a{}_a = h^{ab} K_{ab}$.

Given these quantities, the ADM system of equations can be written as $d(d-1)/2$ evolution equations and d constraint equations for a d-dimensional spacetime. The evolution equations for the induced metric $\mathbf{h}$ are given by

$$\left(\frac{\partial}{\partial t} - L_\beta\right) h_{ij} = -2\alpha K_{ij} \quad (7)$$

and the evolution equations for the extrinsic curvature are given by

$$\left(\frac{\partial}{\partial t} - L_\beta\right) K_{ij} = -D_i D_j \alpha + \alpha \left\{R_{ij} + K K_{ij} - 2K_{ia} K^a_j + 4\pi\left[(S-\rho)h_{ij} - 2S_{ij}\right]\right\} \quad (8)$$

In addition to these evolution equations, the *3 + 1* splitting also places restrictions on the objects that are defined on the initial hypersurface. In other words, one cannot freely specify quantities on the initial hypersurface Σ_0. These imposed constraints come in the form of the energy constraint equation

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \quad (9)$$

and momentum constraint equations

$$D_j K^{ij} - h^{ij} D_j K = 8\pi j^i \quad (10)$$

C. Generalized Harmonic Scheme

The GH formalism is based on coordinates x^μ that are chosen so that each coordinate satisfies a scalar wave equation with source function H^μ: [3]

$$\Box x^\mu = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\mu}\right) = -g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} \equiv H^\mu, \quad (11)$$

where g is the determinant of the metric, and $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. To see why this has proven to be so useful for Cauchy evolution, let us begin by rewriting the field equations in trace-reversed form

$$R_{\mu\nu} = \bar{T}_{\mu\nu}, \quad (12)$$

where

$$\bar{T}_{\mu\nu} = \frac{2}{3}\Lambda_5 g_{\mu\nu} + 8\pi\left(T_{\mu\nu} - \frac{1}{3}T^\alpha{}_\alpha g_{\mu\nu}\right) \quad (13)$$

When viewed as a set of second-order differential equations for the metric $g_{\mu\nu}$, the field equations in the form (12) do not have any well-defined mathematical character (namely hyperbolic, elliptic or parabolic), and in fact are ill-posed. Fixing this character requires choosing a coordinate system. A well-known way to arrive at a set of strongly hyperbolic equations is to impose *harmonic coordinates*, namely (11) with $H^\mu = 0$. Specifically, this condition (and its gradient) can be substituted into the field equations to yield a wave equation for the principal part of each metric element, $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + ... = 0$, where the ellipses denote lower order terms.

One potential problem with harmonic coordinates, in particular in a highly dynamical, strong-field spacetime evolved via a Cauchy scheme, is that beginning from a well-defined initial data surface $t = $ const. which is everywhere space-like, there is no guarantee that t, subject to the harmonic condition, will remain time-like throughout the spacetime as evolution proceeds. If t becomes null or space-like at a point, standard numerical techniques will break down. A solution to this, first suggested in [7], is to make use of source functions (as originally introduced in [8]). Note that *any* spacetime in *any* coordinate system can be written in GH form; the corresponding source functions are simply obtained by evaluating the definition (11). Thus, trivially, if there is a well-behaved, non-singular coordinate chart that covers a given spacetime, then there is a GH description of it. The difficulty in a Cauchy evolution is that this chart is not known *a-priori*, and the source functions H^μ must be treated as independent dynamical fields. Finding a well-behaved coordinate chart then amounts to supplementing the Einstein field equations with a set of evolution equations for H^μ, which can now be considered to encode the coordinate degrees of freedom in our description of the spacetime.

The field equations in GH form thus consist of the Einstein equations (12), brought into hyperbolic form via the imposition of (11)

$$-\frac{1}{2}g^{\alpha\beta}g_{\mu\nu,\alpha\beta} - g^{\alpha\beta}{}_{,(\mu}g_{\nu)\alpha,\beta} - H_{(\mu,\nu)} + H_\alpha\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu} = \frac{2}{3}\Lambda_5 g_{\mu\nu} + 8\pi\left(T_{\mu\nu} - \frac{1}{3}T^\alpha{}_\alpha g_{\mu\nu}\right), \quad (14)$$

together with the relevant evolution equations for the matter, and a set of equations for the source functions, which we write symbolically as

$$\mathcal{L}^\mu[H^\mu] = 0. \quad (15)$$

Even though H^μ are now treated as independent functions, we are only interested in the *subset* of solutions to the expanded system (14),(15) that satisify the GH constraints (11). Introducing

$$C^\mu \equiv H^\mu - \Box x^\mu, \quad (16)$$

we thus seek solutions to (14),(15) for which $C^\mu = 0$. An

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[3] As can be seen from (11) H^μ is not a vector in the sense of its properties under a coordinate transformation, rather it transforms as the trace of the metric connection. One can introduce additional geometric structure in the form of a background metric and connection to write the GH formalism in terms of "standard" tensorial objects. However, in a numerical evolution one must always choose a concrete coordinate system, and hence the resulting equations that are eventually discretized are the same regardless of the extra mathematical structure introduced at the formal level.

equivalent way of obtaining (14) from (12) is to subtract $\nabla_{(\mu}C_{\nu)}$ from $R_{\mu\nu}$, so that

$$R_{\mu\nu} - \nabla_{(\mu}C_{\nu)} - \bar{T}_{\mu\nu} = 0. \tag{17}$$

The effect of this subtraction becomes obvious when we rewrite the Ricci tensor explicitly in terms of $\Box x^\mu$, so $R_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}g_{\mu\nu,\alpha\beta} - g^{\alpha\beta}{}_{,(\mu}g_{\nu)\alpha,\beta} - \nabla_{(\mu}\Box x_{\nu)} - \Gamma^\alpha{}_{\beta\mu}\Gamma^\beta{}_{\alpha\nu}$. We see that the subtraction of $\nabla_{(\mu}C_{\nu)}$ is simply designed to replace the $\nabla_{(\mu}\Box x_{\nu)}$ term in $R_{\mu\nu}$ by an equivalent $\nabla_{(\mu}H_{\nu)}$ term. We also see that a solution of the Einstein field equations (12) is also a solution of (17), as long as the constraints $C^\mu = 0$ are satisfied.

For a Cauchy evolution of the system (14),(15), we need to specify initial data in the form

$$g_{\mu\nu}|_{t=0}, \quad \partial_t g_{\mu\nu}|_{t=0}, \tag{18}$$

subject to the constraints

$$C^\mu|_{t=0} = 0, \quad \partial_t C^\mu|_{t=0} = 0. \tag{19}$$

One can show (see for e.g. [9]) that if (19) is satisfied, then the ADM Hamiltonian and momentum constraints will be satisfied at $t = 0$. Conversely, if the ADM constraints are satisfied at $t = 0$ together with $C^\mu|_{t=0} = 0$ (this latter condition is satisfied trivially computing $H^\mu|_{t=0}$ by substituting (18) into (11)), then $\partial_t C^\mu|_{t=0} = 0$. Thus, our initial data method described in the previous section will produce data consistent with (19). Furthermore, using a contraction of the second Bianchi identity $\nabla^\mu R_{\mu\nu} = \nabla_\nu R/2$, one can show that C^μ satisfies the following hyperbolic equation:

$$\Box C_\nu = -C^\mu \nabla_{(\mu}C_{\nu)} - C^\mu \bar{T}_{\mu\nu}. \tag{20}$$

Thus, if we imagine (analytically) solving (14),(15) using initial data satisfying (19) supplemented with boundary conditions consistent with $C^\mu = 0$ on the boundary for all time, then (20) implies that C^μ will remain zero in the interior for all time.

At the level of the discretized equations, however, C^μ is only zero up to truncation error. This is not *a priori* problematic: numerically one only ever gets a solution approximating the continuum solution to within truncation error. However, experience with asymptotically-flat simulations suggest that in some strong-field spacetimes, equation (20) for C^μ admits exponentially growing solutions (the so-called “constraint-violating modes”). At any practical resolution, this severely limits the amount of physical time for which an accurate solution to the desired $C^\mu = 0$ Einstein equations can be obtained. In asymptotically flat spacetimes, supplementing the GH harmonic equations with *constraint-damping* terms as introduced in [10] suppresses these unwanted solutions. Anticipating similar problems in AAdS spacetimes, and that constraint damping will similarly help, we add the same terms to (14), and arrive at the final form of our evolution equations

$$\begin{aligned} &- \frac{1}{2}g^{\alpha\beta}g_{\mu\nu,\alpha\beta} - g^{\alpha\beta}{}_{,(\mu}g_{\nu)\alpha,\beta} \\ &- H_{(\mu,\nu)} + H_\alpha\Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\mu}\Gamma^\beta{}_{\alpha\nu} \\ &- \kappa\left(2n_{(\mu}C_{\nu)} - (1+P)g_{\mu\nu}n^\alpha C_\alpha\right) \\ &= \frac{2}{3}\Lambda_5 g_{\mu\nu} + 8\pi\left(T_{\mu\nu} - \frac{1}{3}T^\alpha{}_\alpha g_{\mu\nu}\right). \end{aligned} \tag{21}$$

Here, the unit time-like one-form $n_\mu = -\alpha\partial_\mu t$, and the constraint damping parameters $\kappa \in (-\infty, 0]$ and $P \in [-1, 0]$ are arbitrary constants. i

Note that the new terms are homogeneous in C_μ, and hence do not alter any of the properties discussed above that relate solutions of the Einstein evolution equations and ADM constraints with those of the corresponding GH equations, with the exception that the constraint propagation equation (20) picks up additional terms, again homogeneous in C_μ (see for example [10]).

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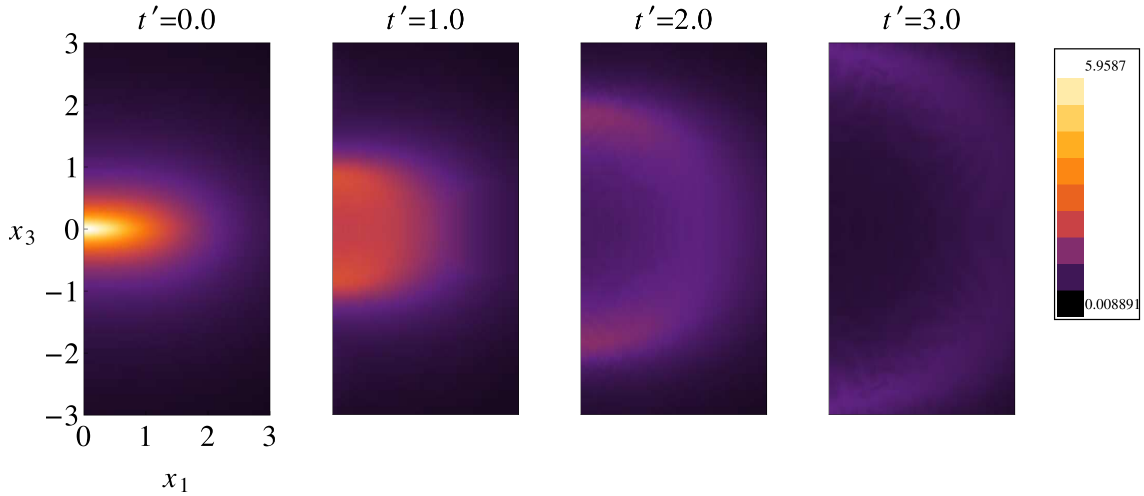

FIG. 1: Temperature $T \sim \epsilon^{(1/4)}$, where ϵ is the energy density of the fluid, for a generalized harmonic simulation run with initial anisotropy in the gravity initial data of $w_y/w_x = 32$, and whose final state black hole has horizon radius $r = r_h = 5$ in global AdS_5 coordinates $(t, r, \chi, \theta, \phi)$. Each plot depicts the spatial profile of temperature in the $x_1 - x_3$ plane of Minkowski space, taken at a constant Minkowski time slice and at an angle of $\phi = 0$. One can recoverthe full spatial dependence by simply rotating each of these $x_1 - x_3$ profiles about the x_3 axis. By interpreting the x_3 axis as the beam-line direction, and x_1 as the radius in the transverse plane, the initial data in the first panel can be thought of as approximating a head-on heavy ion collision at its moment of impact.