

Resummation of non Borel summable series.

Zbigniew Ambroziński*

February 23, 2012

Abstract

This notice is related to the talk given by Mithat Ünsal on 2/3/2012 in KITP. We demonstrate that indeed the ambiguity of Borel sum for non Borel summable system cancels with ambiguity of molecular instanton contribution. In order to do this, we performed high precision computations for the ground energy level of a double well potential in quantum mechanics.

1 Introduction

We consider a double well potential in the following parametrization:

$$V(x) = \frac{1}{2}x^2(1 - \sqrt{g}x)^2. \quad (1)$$

As it is well known (see [1]), the perturbation series for the ground energy $E(g)$ is non Borel summable. Still, one can perform a Borel sum for negative coupling $g < 0$ and then analytically continue it to positive axis going through the lower or upper half of the complex plane. This freedom results in an ambiguity of imaginary part of the Borel sum. However, there is another ambiguity originating from two-instanton molecule contribution to energy. As it is conjectured by Ünsal [2], these two cancel.

To show this fact, we expand the ground energy in powers of g , construct its Borel transform, continue it analytically using Padé approximation and perform its inverse Borel transform. This procedure can be performed only for complex g or in a limit $|g| \pm i\epsilon$. Next, we demonstrate that the two-instanton molecule contribution derived by Bogomolny [3] and then improved by Zinn-Justin and Jentschura [4] has exactly opposite imaginary part. We are able to settle the agreement and to extract three next leading coefficients.

There is yet another, essentially exact numerical technique which we call cut Fock space method that can be used to compute ground energy of the system. It allows us to show that the Borel sum plus two-instanton molecule contribution give a very accurate approximation to the energy for small couplings.

2 Borel sum

In order to perform the Borel summation, we first calculated expansion of the ground state energy up to 350th term $E(g) = \sum_{k=0}^{350} e_k g^k$ (for similar calculation

*Marian Smoluchowski Institute of Physics, Jagiellonian University.
e-mail: zbigniew.ambrozinski@uj.edu.pl

see [5]) and constructed its Borel transform

$$B(t) = \sum_{k=0}^{350} \frac{e_k}{k!} g^k.$$

The Borel transform has a small radius of convergence ($t = 1/3$), so we found its diagonal [175/175] Padé approximant $P(t)$ as a candidate for analytical continuation into a larger domain.

The procedure is then to recover the energy by defining it as an integral

$$E_{Borel}(g) = \frac{1}{g} \int_C dz e^{-z/g} P(z). \quad (2)$$

In a Borel summable case we would have $C = (0, \infty)$. Here however, $P(t)$ has a pole at $t \approx 1/3$ which is the source of non Borel summability. One has then to rotate C into the complex plane, so we have $C = \{e^{i\phi}t : t > 0\}$. The phase ϕ is arbitrary as long as $Re(\phi) > 0$. There is another issue one has to take care of: the Padé approximation is convergent only in some region $|z| < R$. It is then not possible to perform the integral to infinity. Still, for small g the exponent present in the integral dies out very fast, so one gets very precise results integrating only over a finite interval. For our purposes we take $C = \{e^{i\pi/4}t : 0 < t < 2\}$. It cannot be much larger because for this order of Padé approximant the contour C will approach poles of $P(z)$ for $t \approx 3.5$. It means that in that region the Padé approximant breaks down. For large values of g it is needed to obtain more terms e_k and higher order Padé approximant. Details of this analysis will be presented in an upcoming paper [6].

3 Two–instanton molecule

A well known WKB calculation would tell us that there is a contribution to energies of order $e^{-1/6g}$ which comes from instantons. This contribution cancels exactly when one averages the two lowest energies. In [3] Bogomolny stated that there is an attractive interaction between instantons which yields a smaller contribution of order $e^{-1/3g} \log(g)/g$ and is identical for both lowest energies. In [7] the general form of this contribution was postulated:

$$\Delta E_{molecule}(g) = \frac{1}{\pi g} e^{-1/3g} \left[\sum_{l=0}^{\infty} \epsilon_{0l} g^l + \log(-2/g) \sum_{l=0}^{\infty} \epsilon_{1l} g^l \right] \quad (3)$$

with a few first coefficients known:

$$\begin{aligned} \epsilon_{00} &= \gamma, & \epsilon_{10} &= 1, \\ \epsilon_{01} &= -\frac{23}{2} - \frac{53}{6}\gamma, & \epsilon_{11} &= -\frac{53}{6}, \\ \epsilon_{02} &= \frac{13}{2} - \frac{1277}{72}, & \epsilon_{12} &= -\frac{1277}{72}. \end{aligned} \quad (4)$$

Notice that for small g there is $|\log(g)| \gg 1 \gg g|\log(g)| \gg \dots$, so the order of coefficients beginning from the leading one is $\epsilon_{10}, \epsilon_{00}, \epsilon_{11}, \epsilon_{01}, \epsilon_{20}, \epsilon_{21}$. It is important that imaginary part of the logarithm is taken with the plus sign, i.e. $\log(-2/g) = \log(2/g) + i\pi$. It would have been opposite if we had taken the contour C to lie on the lower half of complex plane. In the following we will understand by $\Delta E_{molecule}(g)$ the formula (3) taken with (4) as the leading coefficients and the other terms omitted.

4 Cut Fock space method

A completely alternative method of computing the lowest energies is the cut Fock space method. We denote the basis states of the Fock space, which are eigenstates of the harmonic oscillator, by $|n\rangle$. One can express Hamiltonian $H = \frac{1}{2}P^2 + V(X)$ for the double well potential in terms of creation and annihilation operators. It is then possible to calculate matrix elements $\langle m|H|n\rangle$ of the Hamiltonian. We expect then that eigenvalues of the finite matrix, where $n, m < \text{cut-off}$, will approximate energies of the system. This expectation has been very well confirmed in many cases, see e.g. [8]. We define then

$$E_{Fock}(g) = \frac{1}{2}(E_0 + E_1) \quad (5)$$

where $E_{0,1}$ are the two lowest eigenvalues of the matrix. We average the two lowest energies in order to exclude the one-instanton contributions. The only limits of this method are that the cut-off has to be finite (up to 5000 for purposes of this notice) and precision of computations.

5 Comparison

What we expect is that $\text{Im}E_{Borel}(g) + \text{Im}\Delta E_{molecule}(g) = 0$ and $\text{Re}E_{Borel}(g) + \text{Re}\Delta E_{molecule}(g) = E_{Fock}(g)$ up to terms disregarded in the two-instanton molecule. Energies obtained with both methods are presented in Fig. 1. The six leading order terms of two-instanton molecule were used, so the discrepancy for $g > 0.07$ is caused by higher order terms of the two-instanton molecules rather than four-instanton molecules.

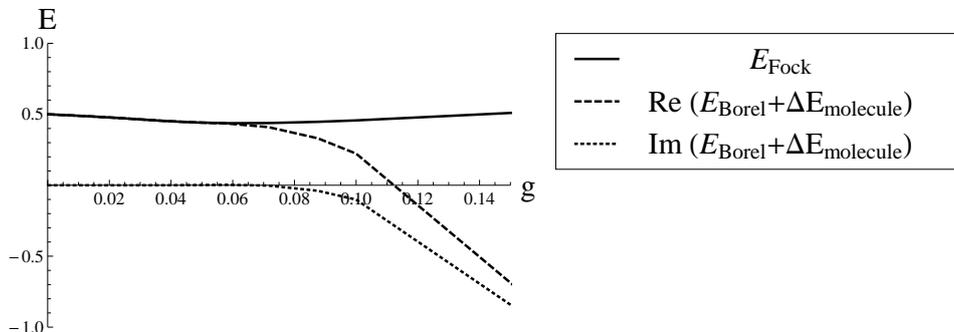


Figure 1: Agreement of E_{Fock} with $E_{Borel} + \Delta E_{molecule}$ is shown. Energies agree for small g . For $g > 0.07$ higher order terms in $\Delta E_{molecule}$ become significant and the agreement is violated.

As the both sides of the latter equation are of order 0.5, it is more instructive to check whether relations

$$\frac{\text{Im}E_{Borel} - \text{Im}\Delta E_{molecule}}{\text{Im}\Delta E_{molecule}} \xrightarrow{g \rightarrow 0} 0, \quad (6)$$

$$\frac{E_{Fock} - \text{Re}E_{Borel} - \text{Re}\Delta E_{molecule}}{\text{Re}\Delta E_{molecule}} \xrightarrow{g \rightarrow 0} 0 \quad (7)$$

hold. We are interested in the limit $g \rightarrow 0$ because we know only a few leading terms of the two-instanton contribution and also because the Borel summation

method is valid for small g . Plots of functions (6) and (7) are presented in Fig. 2. We can see that the convergence is very good. Points plotted in a logarithmic scale lie on a line which slope is around 3. That means that the next correction to the two-instanton molecule should be of order $\exp(-1/3g)g^2$ which is the first term which we did not include.

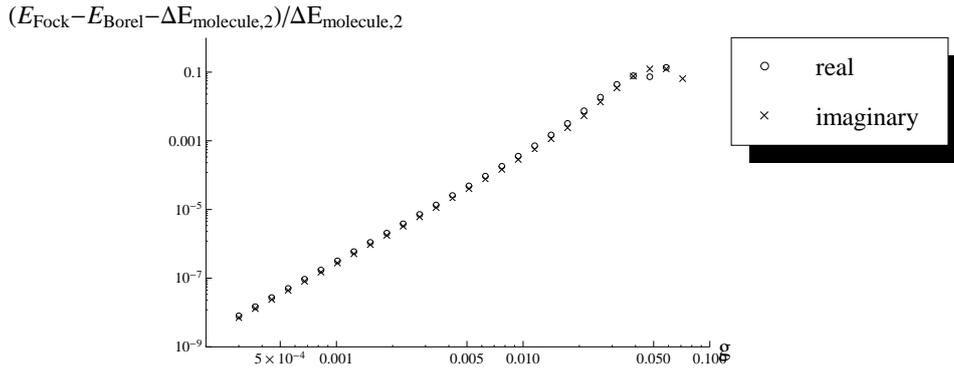


Figure 2: Relative errors of exact value of two instanton contribution and the approximate value $\Delta E_{molecule}$ is presented on this plot. Note that this is a logarithmic plot. Straight line represents power convergence to 0.

6 Summary

It was shown that the ambiguities of the Borel sum and two-instanton molecule indeed cancel as it was postulated. Thanks to high precision of computations it was achievable to confirm values of the coefficients of two-instanton correction given by Zinn-Justin and Jentschura. It should be possible to estimate perhaps more coefficients of the two-instanton molecule straightforward from numerical computations. We shall be aware of limitation of this procedure. Computing energies for smaller values of the coupling constant g would require yet higher precision, e.g. for $g = 10^{-5}$ the instanton correction is of order 10^{-14471} .

Acknowledgements

The author would like to thank J. Wosiek, M. Tepper and Y. Meurice for inspiration and fruitful conversations on this topic.

References

- [1] Jean Zinn-Justin. Multi - instanton contributions in quantum mechanics. *Nucl.Phys.*, B192:125–140, 1981.
- [2] Mithat Unsal. Theta dependence, sign problems and topological interference. 2012.

- [3] E.B. Bogomolny. Calculation of instanton - anti-instanton contributions in quantum mechanics. *Phys.Lett.*, B91:431-435, 1980.
- [4] Ulrich D. Jentschura and Jean Zinn-Justin. Higher order corrections to instantons. *J.Phys.A*, A34:L253-L258, 2001.
- [5] Carl M. Bender and Tai Tsun Wu. Anharmonic oscillator. *Phys.Rev.*, 184:1231-1260, 1969.
- [6] Zbigniew Ambroziński. To be published.
- [7] J. Zinn-Justin and U.D. Jentschura. Multi-instantons and exact results I: Conjectures, WKB expansions, and instanton interactions. *Annals Phys.*, 313:197-267, 2004.
- [8] J. Wosiek. Spectra of supersymmetric Yang-Mills quantum mechanics. *Nucl.Phys.*, B644:85-112, 2002.
Massimo Campostrini and Jacek Wosiek. Exact Witten index in $D = 2$ supersymmetric Yang-Mills quantum mechanics. *Phys.Lett.*, B550:121-127, 2002.
M. Trzetrzelewski and J. Wosiek. Quantum systems in a cut Fock space. *Acta Phys.Polon.*, B35:1615-1624, 2004.